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Anomaly analysis of Hawking radiation from acoustic black hole

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Abstract: The Hawking radiation from the three dimensional rotating acoustic black hole is considered from the viewpoint of anomaly cancellation method initiated by Robinson and Wilczek. Quantum field near the horizon is effectively described by two dimensional charged field with a charge identified as the angular momentum m. The fluxes of charge and energy are obtained from the anomaly cancellation condition and regularity at the horizon, and are shown to match with those of the two dimensional black body radiation at the Hawking temperature.

Keywords: Anomalies in Field and String Theories, Black Holes.

In a space-time background with an event horizon, the quantum effect of fields leads to the radiation known as Hawking radiation [1]. Since Hawking radiation is a key ingredient in the study of black hole physics and in formulating the theory of quantum gravity, it is always worthwhile to have various interpretations for the purpose of deepening the theoretical understanding of it. Recently, as an attempt to understand the nature of Hawking radiation, Robinson and Wilczek [2] proposed a new interpretation that the Hawking radiation plays the role of preserving general covariance at the quantum level by canceling the diffeomorphism anomaly at the event horizon. While there has been a similar work [3] which is specialized to two-dimensional space-time, their proposal is supposed to be valid in any space-time dimension.

The method of anomaly cancellation [2] has been developed by considering the static and spherically symmetric black hole. Its extension to charged and rotating black holes has been done in [4, 5], where the original idea by Robinson and Wilczek has been elaborated. The result of this elaboration is that Hawking radiation is capable of canceling anomalies of local symmetries at the event horizon. In subsequent works [6]–[19],[1] the method of anomaly cancellation have been applied to various kinds of black objects in various dimensions including black ring. Up to now, all the results have obtained correct Hawking fluxes and verified the validity of the method. We note that, as a further elaboration on the method itself, Hawking fluxes of higher-spin currents have been also derived in [21].

In our previous work [14], we applied the anomaly cancellation method to a typical black hole in string theory, the non-extremal D1-D5 black hole [22], and confirmed the idea of Robinson and Wilczek. In this work, we consider a kind of acoustic black hole, the 'draining bathtub' fluid flow referred to as an acoustic analogue of (2+1)-dimensional rotating black hole [23]. The basic idea of acoustic black hole was developed by Unruh [24] in an attempt to connect black hole physics to the theory of supersonic acoustic flows. The acoustic analogy is usually used in the investigation of models proposed for the microphysical origin of Hawking radiation [25]. This naturally leads us to ask whether the idea of anomaly cancellation is valid even for the acoustic black hole. So, in this paper, we would like to study this issue, and eventually it gives a chance to put the connection between the black hole physics and the theory of acoustic flows on a firmer footing.

The acoustic line element of the draining bathtub modeled by a (2+1)-dimensional flow with a sink at the origin is

$$ds^2 = -c^2dt^2 + \left(dr + \frac{A}{r}dt\right)^2 + \left(rd\phi - \frac{B}{r}dt\right)^2 \,, \tag{1}$$

where c is the speed of sound wave, and A and B are arbitrary real positive constants. From this metric, the radial positions of the acoustic even horizon, r_H, and the ergosphere, r_e, are obtained as

$$r_H = \frac{A}{c}\,, \quad r_e = \frac{\sqrt{A^2+B^2}}{c}\,. \tag{2}$$

[1] For a review, see [20]

In dealing with the metric, it is convenient to take a coordinate transformation in the exterior region of $r_H < r < \infty$ [26] given by $dt \to dt + \frac{Ar}{r^2c^2-A^2}dr$, $d\phi \to d\phi + \frac{AB}{r(r^2c^2-A^2)}dr$. The metric (1) is then rewritten in conventional form as

$$ds^2 = -N^2dt^2 + \frac{1}{N^2}dr^2 + r^2(d\phi - \Omega_0 dt)^2\,, \tag{3}$$

where the time coordinate has been rescaled by c for simplicity, and

$$\begin{aligned} N^2(r) &= 1 - \frac{A^2}{c^2r^2} = 1 - \frac{r_H^2}{r^2}\,, \\ \Omega_0(r) &= \frac{B}{cr^2} = \Omega_H\frac{r_H^2}{r^2}\,, \qquad\qquad \Omega_H = \frac{B}{cr_H^2}\,. \end{aligned} \tag{4}$$

Here, Ω_H is the angular speed at the event horizon. The form of the metric (3) looks similar to the rotating BTZ black hole. However, we would like to note that the lapse function $N(r)$ is different from that of the BTZ black hole which is given by $N^2 = (r^2-r_+^2)(r^2-r_-^2)/(r^2l^2)$.

We now consider a real scalar field φ as a test field in the background of eq. (3), and investigate its behavior near the horizon. First of all, the action of φ is evaluated as

$$\begin{aligned} S[\varphi] &= -\int d^3x\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \\ &= \int dtdr\, r\int d\phi\,\varphi\left(-\frac{1}{N^2}(\partial_t + \Omega_0\partial_\phi)^2 + \frac{1}{r}\partial_r rN^2\partial_r + \frac{1}{r^2}\partial_\phi^2\right)\varphi\,. \end{aligned} \tag{5}$$

If we perform a wave decomposition of φ as $\varphi = \sum_m \varphi_m e^{im\phi}$, where m is the angular quantum number and φ_m depends on the coordinates, t and r, then we see that the action is reduced to a two-dimensional effective theory with an infinite collection of fields labeled by m. Next, in order to see what happens near the horizon, it is helpful to take a transformation to the tortoise coordinate r^*, which, in our case, is defined by

$$\frac{\partial r^*}{\partial r} = \frac{1}{N^2} \equiv \frac{1}{f(r)}\,, \tag{6}$$

and leads to $\int dr = \int dr^* f(r(r^*))$. In the near horizon, $f(r(r^*))$ appears to be a suppression factor vanishing exponentially fast, and thus the terms in the action which do not have some factor compensating it can be ignored. In the present case, one can easily see that the last term in the second line of eq. (5) is suppressed, and hence the action near the horizon becomes

$$S[\varphi] = \sum_{m\geq 0}\int dtdr\, r\,\varphi_m^*\left(-\frac{1}{f}(\partial_t + im\Omega_0)^2 + \partial_r f\partial_r\right)\varphi_m\,, \tag{7}$$

where $\varphi_m^* \equiv \varphi_{-m}$. Now it is not so difficult to find that this near horizon action describes an infinite set of massless two-dimensional complex scalar fields in the following background:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{1}{f(r)}dr^2\,, \\ \Phi &= r\,, \quad A_t = -\Omega_0\,, \quad A_r = 0\,, \end{aligned} \tag{8}$$

where Φ is the two-dimensional dilaton field.

Having the two-dimensional effective field theory near the horizon (7) and the two-dimensional background (8), we move on to the problem of Hawking radiation following the approach based on the anomaly cancellation proposed in [2, 4]. The anomaly approach of [2] begins with an observation that, since the horizon is a null hypersurface, all ingoing (left moving) modes at the horizon can not classically affect physics outside the horizon. This implies that they may be taken to be out of concern at the classical level and thus the effective two-dimensional theory becomes chiral, that is, the theory only of outgoing (right moving) modes. If we now perform the path integration of right moving modes, the resulting quantum effective action becomes anomalous under the gauge or the general coordinate transformation, due to the absence of the left moving modes. However, such anomalous behaviors are in contradiction to the fact that the underlying theory is not anomalous. The reason for this is simply that we have ignored the quantum effects of the classically irrelevant left moving modes at the horizon. Thus anomalies must be cancelled by including them.

As stated above, the anomalies are localized at the horizon r_H. However, for actual computation, it is convenient to regard the quantum effective action to be anomalous in an infinitesimal slab, $r_H \leq r \leq r_H + \epsilon$, which is the region near the horizon. The limit $\epsilon \to 0$ will be taken at the end of the calculation. This leads to a splitting of the region outside the horizon, $r_H \leq r \leq \infty$, into two regions, $r_H \leq r \leq r_H + \epsilon$ and $r_H + \epsilon \leq r \leq \infty$. Then we will have the gauge and the gravitational anomaly near the horizon, $r_H \leq r \leq r_H + \epsilon$. The reason for the appearance of the gauge anomaly is that the background (8) contains the effective $\mathrm{U}(1)$ gauge field due to the isometry along the angular direction and the near horizon action (7) is that of charged fields.

We first consider the gauge anomaly. Let us denote the $\mathrm{U}(1)$ gauge current as J_μ. Since the region outside the horizon has been divided into two regions, it is natural to write the gauge current as a sum

$$J^\mu = J^\mu_{(o)}\Theta_+(r) + J^\mu_{(H)}H(r)\,, \tag{9}$$

where $\Theta_+(r) = \Theta(r - r_H - \epsilon)$ and $H(r) = 1 - \Theta_+(r)$. Apart from the near horizon region, the current is conserved

$$\partial_r J^r_{(o)} = 0\ . \tag{10}$$

On the other hand, the current near the horizon is anomalous and obeys the anomalous equation

$$\partial_r J^r_{(H)} = \frac{m^2}{4\pi}\partial_r A_t\,, \tag{11}$$

which is the form of two-dimensional consistent gauge anomaly [27, 28]. Since these two equations in each region are first order differential ones, they can be easily integrated as

$$\begin{aligned} J^r_{(o)} &= c_o, \\ J^r_{(H)} &= c_H + \frac{m^2}{4\pi}\left(A_t(r) - A_t(r_H)\right), \end{aligned} \tag{12}$$

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where c_o and c_H are integration constants. The constant c_o is the electric charge flux which is of our concern.

Now, if W is the quantum effective action of the theory without including the ingoing (left moving) modes near the horizon, then its variation under a gauge transformation with gauge parameter ζ is given by

$$\begin{aligned} -\delta W &= \int d^2x\sqrt{-g}\,\zeta\nabla_\mu J^\mu \\ &= \int d^2x\,\zeta\left[\partial_r\left(\frac{m^2}{4\pi}A_tH\right)+\delta(r-r_H-\epsilon)\left(J^r_{(o)}-J^r_{(H)}+\frac{m^2}{4\pi}A_t\right)\right], \end{aligned} \tag{13}$$

where eqs. (9), (10), and (11) have been used for obtaining the second line. However, as mentioned before, the full quantum effective action of the underlying theory must have gauge invariance. The full effective action includes the quantum effects of the ingoing modes near the horizon, whose gauge variation gives a term canceling the first term of (13). For the gauge invariance, the coefficient of the delta function in eq. (13) should also vanish, and hence, by using eq. (12), we get

$$c_o = c_H - \frac{m^2}{4\pi}A_t(r_H)\ . \tag{14}$$

For determining the charge flux c_o, the value of the current at the horizon, c_H, should be fixed. This is done by imposing a condition that the covariant current [28] given by $\tilde{J}^r = J^r + \frac{m^2}{4\pi}A_t(r)H(r)$ vanishes at the horizon, which, as noted in [5], assures the regularity of physical quantities at the future horizon. Then, the electric charge flux canceling gauge anomaly is determined as

$$c_o = -\frac{m^2}{2\pi}A_t(r_H) = \frac{m^2}{2\pi}\Omega_H = \frac{m^2Bc}{2\pi A^2}\ . \tag{15}$$

The flux of the energy-momentum tensor radiated from the acoustic black hole is similarly determined through the cancellation of the gravitational anomaly. First of all, like the splitting of eq. (9), we write the energy-momentum tensor as

$$T^\mu_\nu = T^\mu_{\nu(o)}\Theta_+(r) + T^\mu_{\nu(H)}H(r)\ . \tag{16}$$

Due to the presence of the gauge potentials and the dilaton in the background (8), the energy-momentum tensor satisfies the modified conservation equation [4]. What is of interest for our problem is the conservation equation for the component T^r_t, the energy-momentum flux in the radial direction. Apart from the near horizon region, it is given by

$$\partial_r T^r_{t(o)} = J^r_{(o)}\partial_r A_t\ . \tag{17}$$

In the near horizon region, we have anomalous conservation equation [4] as

$$\partial_r T^r_{t(H)} = J^r_{(H)}\partial_r A_t + A_t\partial_r J^r_{(H)} + \partial_r N^r_t\ , \tag{18}$$

where $N_t^r = (f'^2 + ff'')/192\pi$ and the prime denotes the derivative with respect to r. The second term comes from the gauge anomaly represented by the anomalous conservation equation, while the third term is due to the gravitational anomaly for the consistent energy-momentum tensor [29]. Now it is not a difficult task to integrate eqs. (17) and (18) and obtain

$$\begin{aligned} T^r_{t(o)} &= a_o + c_o A_t\,, \\ T^r_{t(H)} &= a_H + \int_{r_H}^{r} dr \partial_r \left(c_o A_t + \frac{m^2}{4\pi} A_t^2 + N_t^r \right)\,, \end{aligned} \tag{19}$$

where a_o and a_H are integration constants. Here a_o is the energy flux which we are interested in.

The variation of quantum effective action W under a general coordinate transformation in the time direction with a transformation parameter ξ^t is obtained as

$$\begin{aligned} -\delta W &= \int d^2x \sqrt{-g}\, \xi^t \nabla_\mu T^\mu_t \\ &= \int d^2x\, \xi^t \Bigg[c_o \partial_r A_t + \partial_r \left[\left(\frac{m^2}{4\pi} A_t^2 + N_t^r \right) H \right] \\ &\quad + \left(T^r_{t\,(o)} - T^r_{t\,(H)} + \frac{m^2}{4\pi} A_t^2 + N_t^r \right) \delta(r - r_H - \epsilon) \Bigg]\,. \end{aligned} \tag{20}$$

The first term in the second line is purely the classical effect of the background electric field for constant current flow. The second term is cancelled by including the quantum effect of the ingoing modes as is the case of gauge anomaly. The last term gives non-vanishing contribution at the horizon and is also required to vanish for the general covariance of the full quantum effective action. This requirement leads us to have the following relation.

$$a_o = a_H + \frac{m^2}{4\pi} A_t^2(r_H) - N_t^r(r_H)\,, \tag{21}$$

where the solution (19) has been used. For determining a_o, we first need to know the value of a_H, which is fixed by imposing a condition that the covariant energy-momentum tensor vanishes at the horizon for regularity at the future horizon [5]. Then, from the expression of the covariant energy-momentum tensor [28, 30], $\tilde{T}_t^r = T_t^r + \frac{1}{192\pi}(ff'' - 2(f')^2)$, the condition $\tilde{T}_t^r(r_H) = 0$ gives

$$a_H = \frac{\kappa^2}{24\pi} = 2N_t^r(r_H)\,, \tag{22}$$

where κ is the surface gravity at the horizon,

$$\kappa = \frac{1}{2}\partial_r f|_{r=r_H} = \frac{1}{r_H}\,. \tag{23}$$

Here, from the relation $T_H = \kappa/2\pi$, we see that the Hawking temperature of the acoustic black hole is

$$T_H = \frac{1}{2\pi r_H}\,, \tag{24}$$

which is the correct value [25]. Having the value of a_H, the flux of the energy-momentum tensor is finally determined as

$$\begin{aligned} a_o &= \frac{m^2}{4\pi} A_t^2(r_H) + N_t^r(r_H) \\ &= \frac{m^2}{4\pi}\Omega_H^2 + \frac{\pi}{12} T_H^2 \ . \end{aligned} \tag{25}$$

We have obtained the flux of electric charge, eq. (15), and energy-momentum tensor, eq. (25), via the method of anomaly cancellation. If they are really the Hawking fluxes, they should coincide with the usual thermal fluxes of Hawking (black body) radiation from the black hole. Although the radiation in the case of bosons should be treated, we simply consider the fermion case in order to avoid the superradiance problem. The Hawking distribution for fermions is given by the Planck distribution at the Hawking temperature with electric chemical potential for the charge m of the fields radiated from the black hole,

$$N_m(\omega) = \frac{1}{e^{(\omega - m\Omega_H)/T_H} + 1} \ . \tag{26}$$

Then the electric charge flux of Hawking radiation, say F_a, and the energy-momentum flux of Hawking radiation, say F_E, can be obtained as

$$\begin{aligned} F_a &= m\int_0^\infty \frac{d\omega}{2\pi}(N_m(\omega) - N_{-m}(\omega)) \\ &= \frac{m^2}{2\pi}\Omega_H \ , \end{aligned} \tag{27}$$

$$\begin{aligned} F_E &= \int_0^\infty \frac{d\omega}{2\pi}\omega(N_m(\omega) + N_{-m}(\omega)) \\ &= \frac{m^2}{4\pi}\Omega_H^2 + \frac{\pi}{12}T_H^2 \ . \end{aligned} \tag{28}$$

We see that the thermal fluxes exactly match with eq. (15) and eq. (25). This implies that the fluxes of Hawking radiation from the black hole are capable of canceling the gauge and the gravitational anomalies at the horizon.

In summary, we have obtained the fluxes of Hawking radiation from the three-dimensional acoustic black hole by using the method of anomaly cancellation. It has been shown that the resulting fluxes are precisely the thermal fluxes from the two-dimensional black body radiation at the Hawking temperature. Therefore, our work confirms that the anomaly analysis proposed in [2, 4] is valid for the acoustic geometry modeling the draining bathtub.

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